Chapter 3

The Optical Resonator

3.1 The Plane Mirror Resonator
3.2 The Spherical Mirror Resonator
3.3 Gaussian modes and resonance frequencies
3.4 The Unstable Resonator
Introduction

- The optical resonator (OR) is the optical counterpart of an electronic resonant circuit: it confines and stores light at certain frequencies.

- Most important application: OR as a container within which laser light is generated.

- LASER=OR containing a light amplifying medium

- OR determines frequency and spatial distribution of the laser beam
Introduction

From Fundamentals of Photonics, Saleh and Teich, Wiley, chap 10, p.366
Introduction

- Mirror resonators: 2 or 3 mirrors, 2D or 3D cavities
- Dielectric Resonators: use TIR instead of mirrors:
  - Fiber rings and integrated optic rings
  - Microdisks, microspheres, etc (Whispering Gallery modes)
  - Micropillars
  - Photonic Crystals
- Currently: Nanolasers with quantum confinement of carriers (e.g. electrons) or photons
Introduction

- Two key parameters:
  - Modal volume $V$: volume occupied by confined optical mode
  - Quality factor $Q$: proportional to storage time in units of optical period

- $V$ and $Q$ represent the degrees of spatial and temporal confinements, respectively

- Large $Q$ means low-loss resonator
3.1- Plane Mirror Resonator

- Fabry-Perot interferometer: pair of plane mirrors separated by distance \( d \).

Monochromatic plane: \( u(\vec{r}) = \text{Re}[U(\vec{r})e^{2\pi i \nu t}] \)

Satisfies Helmholtz equation:

\[
\nabla^2 U + k^2 U = 0 \quad \text{with} \quad k = 2\pi \frac{\nu}{c}
\]
3.1- Plane Mirror Resonator

- Standing wave solution is obtained for the boundary conditions:

\[ U(\vec{r}) = 0 \text{ at } z = 0 \text{ and } U(\vec{r}) = 0 \text{ at } z = d \]

\[ U(\vec{r}) = A \sin k z \text{ with } kd = q\pi \Rightarrow k = q \frac{\pi}{d}, \; q = 1, 2, 3, \ldots \]

- \( q \) is the mode number, mode frequencies are:

\[ \nu_q = q \frac{c}{2d} \]

- Arbitrary wave = superposition of modes

\[ U(r) = \sum_q A_q \sin k_q z \]

- Constant frequency difference between adjacent modes (free spectral range): \( \nu_F = \frac{c}{2d} \)
3.1- Plane Mirror Resonator

- The resonance wavelengths in the optical medium are: \( \lambda_q = \frac{c}{\nu_q} \implies 2d = q\lambda_q \)

- Examples:
  - \( d = 30 \text{ cm}, n = 1 \) (air), free spectral range = 500 MHz
  - \( d = 3 \text{ microns}, n = 1 \) (air), 50 THz (7 modes in visible range: \( q = 8, \ldots, 14, \lambda_q = 750, \ldots, 429 \text{ nm} \))
  - Free spectral range can be adjusted by placing resonators in series
3.1- Plane Mirror Resonator

- Calculation of light intensity in the resonator:
  - Summation of multiply reflected amplitudes
  - Phase shift after one round trip of propagation (2d) is \( \varphi = \frac{2\pi}{\lambda} 2d = 2kd \)
  - Wave reproduces itself after a round trip, thus: \( \frac{2\pi}{\lambda} 2d = 2kd = 2q\pi, \ q = 1, 2, 3... \)

\[ U(r,t) = U_0 e^{i(k.r - 2\pi vt)}, \ k. r = \frac{2\pi}{\lambda} 2d \ for \ r = 2d \]
3.1- Plane Mirror Resonator

Calculation of light intensity in the resonator:

- Summation of multiply reflected amplitudes for perfect "non lossy" resonator:

\[
U_1 = U_0 e^{-i \frac{2\pi}{\lambda} 2d}, \\
U_2 = U_1 e^{-i \frac{2\pi}{\lambda} 2d} = U_0 e^{-i \frac{2\pi}{\lambda} 4d}, \\
\text{etc...} \\
U_j = U_0 e^{i \left( \frac{2\pi}{\lambda} \right) 2jd}
\]

Total amplitude:

\[
U = U_0 + U_1 + U_2 + \ldots = U_0 \sum_{j=0}^{N} e^{i \left( \frac{2\pi}{\lambda} \right) 2jd}
\]
3.1-Plane Mirror Resonator

- If resonator has losses, amplitude reduction upon reflection is taken into account (r reflection coefficient):

Total amplitude: \( U = U_0 + rU_1 + r^2U_2 + ... \)

\( r = \text{complex reflection coefficient (overall amplitude attenuation)} \)

Intensity: \( I = |U|^2 \)

\[
I = \frac{I_0}{1 + \left(\frac{2F}{\pi}\right)^2 \sin^2 \left(\frac{\varphi}{2}\right)} = \text{transmitted intensity}
\]

\[
I_0 = |U_0|^2 = \text{incident intensity}
\]

\( R = |r|^2, \text{reflectivity of lossy mirror (or overall losses over round trip)} \)

\[
F = \frac{\sqrt{|r|}}{1 - |r|} \quad \text{Finesse of resonator} = \frac{\text{Intermode spacing}}{\text{Width of a mode}} = \frac{\nu_F}{\delta \nu}
\]

\[
U_{\text{Total}} = U = U_0 \sum_{j=0}^{N} r^j e^{i \left(\frac{2\pi}{\lambda}\right)2jd}
\]
3.1- Plane Mirror Resonator

Spectral width depends strongly on finesse \( F \). Cf. \( F=100 \) vs 10…

Spectral Response of the lossy resonator:

\[
\frac{I}{I_0} = \frac{1}{1 + \left(\frac{2F}{\pi}\right)^2 \sin^2 \left(\pi \frac{\nu}{\nu_F}\right)}
\]

\( \nu_F = \frac{c}{2d}, I_0 \equiv \text{Incident intensity} \)

\( \nu_F = \text{round-trip time} \)

\( \frac{1}{\nu_F} \approx \frac{\nu_F}{F} \)

Spectral width of a mode \( \approx \frac{\nu_F}{F} \)

\[
y = \frac{1}{1 + 100 \sin^2 (100\pi x)}
\]

\[
y = \frac{1}{1 + 10 \sin^2 (100\pi x)}
\]
3.1- Plane Mirror Resonator

- The two principal sources of loss in the optical resonator are
  - Absorption and scattering in the medium between the mirror (see laser amplifier):

  \[
  \text{Round trip power attenuation: } \exp(-2\alpha_s d)
  \]

  \(\alpha_s\) : linear absorption coefficient of the medium

  - Losses arising from imperfect reflection at the mirrors (necessary transmission + finite size effects):

  Mirrors of reflectance: \(R_1\) and \(R_2\)

  Overall round trip loss of intensity:

  \[
  R_1R_2 \exp(-2\alpha_s d) \equiv r^2
  \]

  Overall distributed-loss written as:

  \[
  \exp(-2\alpha_r d) = R_1R_2 \exp(-2\alpha_s d)
  \]

  \[
  \alpha_r = \alpha_s + \frac{1}{2d} \log \left( \frac{1}{R_1R_2} \right)
  \]

  Ultimately (after maths):

  \[
  F \approx \frac{\pi}{\alpha_r d}
  \]

  if \(\alpha_r d \ll 1\) (small losses)
3.1 Plane Mirror Resonator

- The resonance linewidth is inversely proportional to the loss factor ($\alpha_r d$)

Finesse is by definition $F = \frac{v_F}{\delta v} \Rightarrow \delta v \approx \frac{c}{2d} \frac{\pi}{\alpha_r d} = \frac{c\alpha_r}{2\pi}$

$\alpha_r$ is the loss per unit length, $c\alpha_r$ is the loss per unit time

The resonator lifetime or photon lifetime in cavity is:

$\tau_p = \frac{1}{c\alpha_r}$, thus $\delta v = \frac{1}{2\pi \tau_p}$
3.1 Plane Mirror Resonator

- The Quality factor $Q$ can be used to characterise the losses:

$$Q = 2\pi \frac{\text{Stored energy}}{\text{Energy loss per cycle}}$$

In the case of an optical resonator (laser), one can show that:

$$Q = 2\pi \nu_0 \tau_p$$

$$Q = \frac{\nu_0}{\nu_F} F, \nu_0 = \text{frequency of one of the modes}$$

- Since the resonator frequencies are much larger than the mode spacing, then $Q \gg F$
3.1-Plane Mirror Resonator

- What are the requirements for a laser:
  - Assume 3-D resonator (3 pairs of parallel mirrors, closed resonator), equivalent to black-body cavity.
  - Number and frequency of modes is given by the particle in the box model (photons):

\[
\frac{dN}{V} = \frac{8\pi v^2}{c^3} \, dv
\]

\[V = 1cm^3, \nu = 3 \times 10^{14} \text{ Hz}, \, d\nu = 3 \times 10^{10} \text{ Hz}\]

\[dN = 2 \times 10^9 \text{ modes}\]
3.1-Plane Mirror Resonator

- All the modes would have comparable Q in the 3D resonator
  - To be avoided in a laser as it would cause all the atoms to emit power into a large number of modes (would differ in their frequency and spatial characteristics)

- Large, open resonators consisting of opposite flat/curved reflectors must be used:
  - Energy of the vast majority of modes lost after a single pass
  - Surviving modes are near the axis
3.2 Spherical Mirror Resonator

- Ray confinement:
  - Concave $R<0$,
  - Convex $R>0$,
  - Only meridional (lie in a plane passing through the optical axis) and paraxial rays are considered

- Geometric optics is sufficient to find the condition for the existence of the confined modes

$$g_1 = 1 + \frac{d}{R_1}, \quad g_2 = 1 + \frac{d}{R_2}$$

3.2 Spherical Mirror Resonator

- Condition for the existence of confined modes:
  - Outside this domain the resonator is said to be unstable
    \[0 \leq g_1 g_2 \leq 1\]

- For same radii, stability condition becomes:

  \[R_1 = R_2 \Rightarrow g_1 = g_2 = g\]
  
  \[-1 \leq g \leq +1 \Rightarrow 0 \leq \frac{d}{(-R)} \leq 2\]

- Three resonators of practical interest: confocal concentric, confocal/planar
3.2-Spherical Mirror Resonator

Stability condition:

\[ R_1 = R_2 \Rightarrow g_1 = g_2 = g \]

\[ -1 \leq g \leq +1 \Rightarrow 0 \leq \frac{d}{(-R)} \leq 2 \]

- a. Planar \((R_1 = R_2 = \infty)\)
- b. Symmetrical confocal \((R_1 = R_2 = -d)\)
- c. Symmetrical concentric \((R_1 = R_2 = -d/2)\)
- d. Confocal/planar \((R_1 = -d, R_2 = \infty)\)
- e. Concave/convex \((R_1 < 0, R_2 > 0)\)
3.3 Gaussian Modes and resonance frequencies

- Gaussian beams are stable modes of the spherical mirror resonator: wavefronts and phase match exactly the boundary conditions imposed by spherical mirror resonator (Helmholtz paraxial equation).

- Gaussian beam retraces incident beam if the radius of the wavefronts is exactly the same as the mirror radius.

- Phase of Gaussian beam:

\[ \varphi(R, z) = kz - \xi(z) + \frac{k\rho^2}{2R(z)} \text{ with } \rho^2 = x^2 + y^2 \]

On-axis: \[ \varphi(0, z) = kz - \xi(z) \]

\[ \xi(z) = \arctan \left( \frac{z}{z_0} \right) : \text{phase retardation with respect to plane wave} \]
3.3 Gaussian Modes and resonance frequencies

- See chapter 10 for details (symmetrical resonator):

\[ R_1 = R_2 = -|R| \]
\[ z_1 = -d/2, \quad z_2 = d/2 \]
\[ R(z) = z \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right] \]
\[ W(z) = W_0 \left[ 1 + \left( \frac{z}{z_0} \right)^2 \right]^{1/2} \]

\[ z_0 = \frac{d}{2} \left( 2 \frac{|R|}{d} - 1 \right)^{1/2} \]
\[ W_0^2 = \frac{\lambda d}{2\pi} \left( 2 \frac{|R|}{d} - 1 \right)^{1/2} \]
\[ W_1^2 = W_2^2 = \frac{\lambda d}{\pi} \left\{ \left( \frac{d}{|R|} \right) \left[ 2 - \left( \frac{d}{|R|} \right) \right] \right\}^{1/2} \]
3.3 Gaussian Modes and resonance frequencies

\[ z_2 = z_1 + d \]

\[ R(z) = z + \frac{z_0^2}{z} \]

\[ R_1 = z_1 + \frac{z_0^2}{z_1}, \quad -R_2 = z_2 + \frac{z_0^2}{z_2} \]

\[ z_1 = \frac{-d(R_2 + d)}{R_2 + R_1 + 2d} \]

\[ z_0^2 = \frac{-d(R_1 + d)(R_2 + d)(R_1 + R_2 + d)}{(R_2 + R_1 + 2d)^2} \]
3.3 Gaussian Modes and resonance frequencies

- See chapter 10 for details (symmetrical resonator):

\[ R_1 = R_2 = -|R| \]
\[ z_1 = -d/2, \quad z_2 = d/2 \]

\[ W(z) = W_0 \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right]^{1/2} \]

\[ W(z) = W_0 \left[ 1 + \left( \frac{z}{24z_0} \right)^2 \right]^{1/2} \]
3.3 Gaussian modes and resonance frequencies

- Resonance frequencies can be calculated from the resonance condition (round trip phase change is exactly $2\pi$):

At mirrors location:
\[ \phi(0, z_1) = kz_1 - \xi(z_1) \text{ and } \phi(0, z_2) = kz_2 - \xi(z_2) \]

Phase change from $z1$ to $z2$:
\[ \Delta \phi = \phi(0, z_2) - \phi(0, z_1) = k(z_2 - z_1) - [\xi(z_2) - \xi(z_1)] = kd - \Delta \xi \]

For one round trip + phase matching condition:
\[ \Delta \phi = 2(kd - \Delta \xi) = 2q\pi \ (q = \pm 1, \pm 2,...) \]

\[ \nu_q = q\nu_F + \frac{\Delta \xi}{\pi} \nu_F, \text{ frequency spacing: } \left( \nu_F = \frac{c}{2d} \right) \]
3.3 Gaussian modes and resonance frequencies

- All the Hermite-Gaussian beams of order \((l,m)\) are also good solutions.

- All \((l,m)\) modes have same wavefronts as \((0,0)\) but different amplitudes: Conditions for wavefront matching are identical.

- The entire family of \(A_{l,m} G_l G_m\) are also modes of the spherical mirror resonator.

- The resonance frequencies depend on \((l,m)\)
Gaussian Beams - Modes

Intensity distribution of Hermite-Gaussian modes:

\[ I_{lm}(x,y,z) = |A_{lm}|^2 \left[ \frac{W_0}{W(z)} \right]^2 G_l^2 \left( \frac{\sqrt{2}x}{W(z)} \right) G_m^2 \left( \frac{\sqrt{2}y}{W(z)} \right) \]

TEM_{lm} modes: \( G_l, G_m \) Hermite-Gaussian function of order \( l, m \)

\( A_{lm} = \text{constant (} l, m \) \)

TEM_{00} = Gaussian Beam
3.3 Gaussian modes and resonance frequencies

- Phase matching conditions provide resonance frequencies:

  Phase of the axial modes:
  
  \[ \varphi(0,z) = kz - (l + m + 1)\xi(z) \]

  After a round trip + phase matching condition

  \[ 2kd - 2(l + m + 1)\Delta\xi = 2q\pi \quad (q = \pm1, \pm2, \ldots) \]

  Resonance frequencies:

  \[ \nu_q = q\nu_F + (l + m + 1)\frac{\Delta\xi}{\pi}\nu_F \]

- Modes of different \( q \) but same \((l,m)\) are called **longitudinal (axial)** modes

- Modes with different \((l,m)\) represent different **transverse** modes
3.4 Unstable Resonator

- Close to regions of ‘unconfinement’, beam size increases
- Light losses due to missing the mirror become important (diffraction losses).
- For high power applications, large volume modes and diffraction losses are desirable
- High diffraction losses are good for a high gain situation (see later).
- Output beam has large aperture: optics are simplified
- Losses depend only on mirrors radii of curvature and separation distance.
3.4 Unstable Resonator

- Spherical wave picture of the mode in an unstable resonator.
- Points $P_1$ and $P_2$ are the virtual centres of the spherical waves.